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Computing a Minimum Weight Triangulation of a Sparse Point Set*

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Abstract. Investigating the minimum weight triangulation of a point set with constraint is an important approach for seeking the ultimate solution of the minimum weight triangulation problem. In this paper, we consider the minimum weight triangulation of a *sparse point set*, and present an $O(n^4)$ algorithm to compute a triangulation of such a set. The property of sparse point set can be converted into a new sufficient condition for finding subgraphs of the minimum weight triangulation. A special point set is exhibited to show that our new subgraph of minimum weight triangulation cannot be found by any currently known methods.

Key words: Algorithm, Computational geometry, Minimum weight triangulation

1. Introduction

Let $S = \{p_i \mid i = 0, \ldots, n-1\}$ be a set of n points in the plane, where each point p_i has the coordinates $(x(p_i), y(p_i))$. For simplicity, we assume that S is in general position so that no three points in S are co-linear. Let $\overline{p_i p_j}$ for $i \neq j$ denote the line segment with endpoints p_i and p_j, and let $\omega(p_i p_j)$ denote the weight of $\overline{p_i p_j}$, that is the Euclidean distance between p_i and p_j.

A *triangulation* of a planar point set S, denoted by $T(S)$, is a maximum set of line segments with endpoints in S in which no two line segments share any interior point of them, thus $T(S)$ partitions the interior of the convex hull of S into empty triangles. The weight of a triangulation $T(S)$ is given by

$$\omega(T(S)) = \sum_{\overline{p_i p_j} \in T(S)} \omega(p_i p_j).$$

A *minimum weight triangulation*, simply MWT, of S is defined as

$$MWT(S) = \min\{\omega(T(S)) \mid \text{for all possible } T(S)\}.$$

*This work was completed while the second author visited the Department of Computer Science, Memorial University of Newfoundland.

Computing an $MWT(S)$ is an outstanding open problem whose complexity status is unknown [10, 17]. A great deal of work has been done to seek the ultimate solution of the problem. Basically, there are two directions from which to attack the problem. The first one is to identify the edges inclusive or exclusive to $MWT(S)$ [5, 7, 12, 21]. It is obvious that the intersection of all triangulations of S is a subset of $MWT(S)$. Recently, Dickerson and Montague [7] observed that the intersection of all local optimal triangulations of S is a subgraph of $MWT(S)$. A triangulation $T(S)$ is called k-gon local optimal, denoted by $T_k(S)$, if any k-gon attracted from $T(S)$ is optimally triangulated by the edges of $T(S)$. If the $MWT(S)$ is unique, then the following inclusion property holds:

$$\bigcap T(S) \subseteq \bigcap T_4(S) \subseteq \cdots \subseteq \bigcap T_{n-1}(S) \subseteq MWT(S).$$

However, it seems difficult to find the intersection as k increases, and so far only a subgraph of $T_4(S)$ has been found by [7]. Gilbert [9] showed that the shortest edge in S is in $MWT(S)$. Yang et al. [21] showed that mutual nearest neighbors in S are also in $MWT(S)$. Keil [12] showed that the so-called β-skeleton of S for $\beta = \sqrt{2}$ is a subgraph of $MWT(S)$. Cheng and Xu [5] extended Keil's result to $\beta = 1.17682$. It seems that the identification of more edges in $MWT(S)$ is a promising approach. This is because the more the edges of $MWT(S)$ are identified, the less disjoint connected components in $MWT(S)$ will be. It is possible that eventually all these identified edges form a connected graph so that an $MWT(S)$ can be constructed by a dynamic programming method in polynomial time. Moreover, even if such a connected graph is impossible to obtain, a larger subgraph will lead to a better performance by some heuristics [20].

The other direction is to construct exact $MWT(S)$ with some constraint on S. Gilbert [9] and Klinesek [13] investigated the case where S is restricted to a simple polygon. An $O(n^3)$ time dynamic programming algorithm was proposed to obtain an $MWT(S)$. Anagnostou and Corneil [1] studied the situation where S is restricted to k nested convex polygons. They gave an $O(n^{3k+1})$ time algorithm to find an $MWT(S)$. Meijer and Rappaport [15] later improved the time bound to $O(n^k)$ when S is restricted to k non-intersecting line segments inside the convex hull of S. Cheng et al. [6] and Xu [18] showed that if a subgraph of $MWT(S)$ with k connected components is known, then the complete $MWT(S)$ can be computed in $O(n^{k+2})$ time. In addition to the potential applications of constraint cases, it is hoped that the research on constraint cases would reveal some insight to the solution for the general case.

In this paper, we investigate the situation that S forms a sparse set, which informally speaking, has a property that the distance between two consecutive convex layers of the set is longer than the diameter of the inner layer. We present an $O(n^4)$ time algorithm for computing an $MWT(S)$ for a sparse set S. Amazingly, unlike the most known constrained MWT algorithms which depend on the number of disjoint connected components, the time complexity of our algorithm is independent on the number of convex layers k. Furthermore, we can convert the property of sparse set to a new sufficient condition for finding subgraphs of an $MWT(S)$. By demonstrating some special point set, we show that our new subgraphs cannot be found by any currently known methods [5, 7, 9, 12, 18].

The paper is organized as follows. In Section 2, we discuss some properties of a point set restricted to its convex layers. In Section 3, we present an algorithm that produces an $MWT(S)$ with convex layers constraint. In Section 4, we define a sparse point set S and propose an $O(n^4)$ algorithm to compute an $MWT(S)$. In Section 5, we make some concluding remarks. In particular, we describe a sufficient condition for some edges to be in $MWT(S)$ and also demonstrate a point set whose new subgraph of MWT cannot be found by any known method.

2. Notations and lemmas

The convex layers of a set S of points in the plane, denoted by $CL(S)$, is the set of nested convex polygons obtained by repeatedly computing the convex hull of the remaining set after removing the vertices of the current convex hull. Computing the convex layers of a planar point set was discussed in many papers [3, 16]. An optimal $\Theta(n \log n)$ time algorithm was given by Chazelle [3].

FACT 1. (3). *Convex layers $CL(S)$ for $|S| = n$ can be found in $O(n \log n)$ time and $O(n)$ space.*

Let $CL(S) = (C_1, C_2, \ldots, C_k)$ be the convex layers of S, where C_i for $i = 1, \ldots, k$ is the ith layer of S. Let $V(C_i)$ be the vertex set of C_i and let $|V(C_i)| = n_i$. Let $R(C_i)$ be the interior region bounded by C_i and let $R_{i,i+1}$ denote the region between $R(C_i)$ and $R(C_{i+1})$.

LEMMA 1. *Let $CL(S) = (C_1, \ldots, C_k)$ and let $T_{CL}(S)$ be any triangulation with $CL(S)$ constraint. For each vertex p of C_i, there exists a vertex q of C_{i-1} such that edge $\overline{pq}$ belongs to $T_{CL}(S)$.*

Proof. Since p is a vertex of C_i for $1 < i \leqslant k$, p is an interior point of $R(C_{i-1})$. Since the inner angle at the shared endpoint p of any two consecutive edges of C_i is less than π, there must exist an edge $e \epsilon T_{CL}(S)$ lying on $R_{i-1,i}$ and $e = \overline{pq}$ for $q \epsilon V(C_{i-1})$. □

Let $MWT_{CL}(S)$ denote the minimum weight triangulation of S with convex layers constraint. Figure 1 shows an $MWT(S)$ and an $MWT_{CL}(S)$ for a given point set S.

3. The algorithm for computing an $MWT_{CL}(S)$

Let $T_{CL}(S)$ be any triangulation of S with $CL(S) \in T_{CL}(S)$, and let $\omega(T_{CL}(S))$ be its weight. A minimum weight triangulation with convex layers constraint, $MWT_{CL}(S)$, is one which minimizes $\omega(T_{CL}(S))$ among all possible $T_{CL}(S)$. It

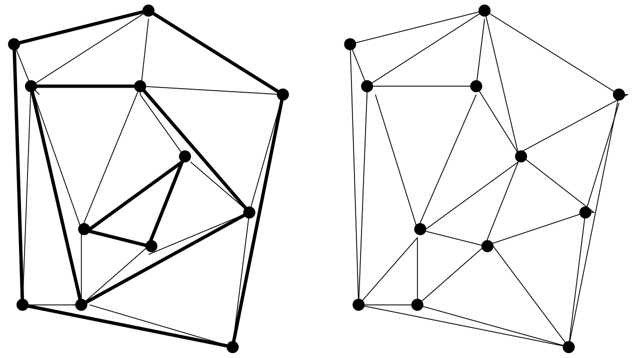

Figure 1. The left-hand side is $MWT_{CL}(S)$ and the right-hand side is $MWT(S)$.

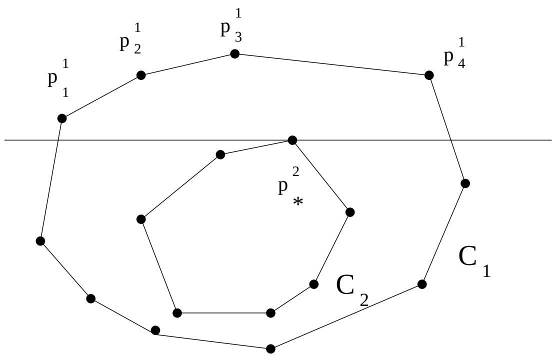

Figure 2. For the definition of p^i_*.

is obvious that to find an $MWT_{CL}(S)$ is easier than to find an $MWT(S)$. This is because the convex layers $CL(S)$ are already known to be a subset of $MWT_{CL}(S)$, a polynomial time algorithm for computing an $MWT_{CL}(S)$ is possible.

FACT 2. (9,14,18). *If L is a set of non-intersecting edges with endpoints in S such that $G(S, L)$ is a planar connected graph, then an MWT of S with L constraint, denoted by $MWT_L(S)$, can be found in $O(n^3)$ time for $|S| = n$.*

Xu [18] analyzed the optimal cell triangulation algorithm given by Heath and Pemmarajiu [11] and obtained an $O(n^3)$ algorithm for computing an $MWT_L(S)$, where L is a subset of non-intersecting edges with endpoints in S and $G(S, L)$ is a planar connected graph. We denote this algorithm as $\mathbf{A - T_L}$.

Since $MWT_{CL}(S)$ only minimizes the total weight of edges between convex layers, we first consider how to triangulate region $R_{1,2}$ so that the total weight of edges in $R_{1,2}$ is a minimum. Let p^2_* be the vertex in C_2 with the maximum y-coordinate (for convenience, we can assume that no two points in S have the same y-coordinate), and let $N(p^2_*)$ be the subset of vertices of C_1 whose y-coordinates are greater than that of p^2_*, i.e., $y(q) > y(p^2_*)$ for $q \in N(p^2_*)$. Figure 2 shows the definition of p^2_* and $N(p^2_*)$, where $N(p^2_*) = (p^1_1, p^1_2, p^1_3, p^1_4)$. By Lemma 1, there exists at least one vertex $p_*^1 \in N(p_*^2)$ such that edge $\overline{p_*^2 p_*^1}$ is in an $MWT_{CL}(S)$. In order to identify such an edge, we have to check all possible edges ending at p_*^2 and $N(p_*^2)$ and their corresponding constraint MWTs. Vertex p_*^2 can be easily found in at most $O(n_2)$ time by scanning the y-coordinates of the vertices of C_2, $N(p_*^2)$ can be computed in at most $O(n_1)$ time by scanning these vertices of C_1 with y-coordinates greater than $y(p_*^2)$. For each vertex $q \epsilon N(p_*^2)$, add edge $\overline{qp_*^2}$ to form a graph $G(V(C_1) \cup V(C_2), C_1 \cup C_2 \cup \{\overline{qp_*^2}\})$. Clearly, the graph G is planar and connected. By Fact 2, an $MWT(V(C_1) \cup V(C_2))$ with $L(= C_1 \cup C_2 \cup \{\overline{qp_*^2}\})$ constraint can be found in $O((n_1 + n_2)^3)$ time by algorithm $\mathbf{A} - \mathbf{T_L}$. Then, an $MWT(V(C_1) \cup V(C_2))$ with $C_1 \cup C_2$ constraint can be found in at most $O(| N(p_*^2) | (n_1 + n_2)^3)$ time.

In the following, we describe an algorithm, denoted by $A - MWT_{CL}$, to produce an MWT of S with convex layers constraint.

Let $CL(S) = (C_1, \ldots, C_k)$, and let p_*^i denote the vertex of C_i with maximal y-coordinate. Let N(p_*^i) denote those vertices of C_{i-1} whose y-coordinates are greater than that of p_*^i.

ALGORITHM $\boldsymbol{A - MWT_{CL}}$

Input: S (a set of n points in general position).

Output: $MWT_{CL}(S)$

1. Find the convex layers $CL(S) = (C_1, \ldots, C_k)$.
2. $For\ i = 2$ to $k\ Do$
   - (a) Find p_*^i and $N(p_*^i)$.
   - (b) *While* $N(p_*^i) \neq \emptyset$ *Do*
     - (i) $q \leftarrow$ *attract*$(N(p_*^i))$;
     - (ii) Compute an $MWT_{C_i \cup C_{i-1} \cup \{\overline{p_*^i q}\}}(V(C_i) \cup V(C_{i-1}))$ by $\mathbf{A} - \mathbf{T_L}$;
     - (iii) Update the minimum $MWT_{C_i \cup C_{i-1}}(V(C_i) \cup V(C_{i-1}))$;
     - (iv) *EndWhile*
   - (c) *EndDo*
3. Produce $MWT_{CL}(S)$ by combining $MWT_{C_i \cup C_{i-1}}(V(C_i) \cup V(C_{i-1}))$ for all $i \in [2, k]$.

The correctness and the time complexity of algorithm $\mathbf{A} - \mathbf{MWT_{CL}}$ are shown as follows.

THEOREM 1. *An $MWT_{CL}(S)$ can be found in O(n^4) time, where S is a set of n points in general position.*

Proof. We apply $\mathbf{A} - \mathbf{MWT_{CL}}$ to S, which correctly computes an $MWT_{CL}(S)$ since $\mathbf{A} - \mathbf{T_L}$ correctly computes an $MWT_{C_i \cup C_{i-1} \cup \{\overline{p_*^i q}\}}(V(C_i) \cup V(C_{i-1}))$. Consider the time complexity. Step 1 can be done in $O(n \log n)$ time by Fact 1. Step 2 executes $k(= O(n))$ times, where Step (a) takes $O(n)$ time in the entire Step 2. By Fact 2, an $MWT_{C_1 \cup C_2}(R_{i-1,i})$ can be found in at most $O((n_{i-1} + n_i)^3)$ time for $i = 2, \ldots, k$. Thus, Step (b) takes $O(n_{i-1} + n_i)^3 * N(p_*^i))$ time. Since the process ends at finding an $MWT(R_{k-1,k})$, then the total computation in Step 2 is at most

$$\sum_{i=2}^{k} O(|\ N(p_*^i)\ |\ (n_{i-1}+n_i)^3) \leqslant O(n^3) \sum_{i=2}^{k} |\ N(p_*^i)\ | \leqslant O(n^4).$$

Step 3 takes $O(n)$ time. □

4. Computing an MWT of a sparse set

We now show that when S is a 'sparse point set', then $MWT_{CL}(S)$ is an $MWT(S)$. We shall first give a fact and some definitions.

FACT 3. *Let P be a simple polygon with n vertices, and let $T(P)$ and $T'(P)$ be any two triangulations of the interior of P, respectively. Then, the number of interior edges of $T(P)$ is equal to that of $T'(P)$, which is $n-3$.*

Proof. By Euler's formula of $e = v + f - 2$ and by the fact that the interior of a simple polygon is triangulated, we have exact $n-3$ triangles in any triangulation of P. Note that each of the n boundary edges corresponds to a triangle and each interior edge is shared by exactly two triangles. Then we have $n-3$ interior edges for any triangulation of a simple polygon with n vertices. □

DEFINITION 1. *The* **diameter** *of a point set S, denoted by $D(S)$, is the maximum Euclidean distance among the pairs of points in S.*

DEFINITION 2. *The* **minimum set distance** *of two point sets S_1 and S_2, denoted by $d(S_1, S_2)$, is the minimum Euclidean distance between the points of S_1 and the points of S_2.*

DEFINITION 3. *Let $CL(S) = (C_1, \dots, C_k)$ be the convex layers of a point set S. S is called* **sparse** *if it satisfies the following two conditions:*

(i) $d(V(C_i), V(C_{i+1})) \geqslant D(V(C_{i+1}))$, for all $i = 1, \cdots k-1$, and
(ii) if edge $\overline{p_j^i p_{j+1}^i}$ of C_i crosses $\overline{pq}$ for $p, q \in S$, $q \in C_l$, and $l < i$, then $d(p,q) > \max\{d(q, p_j), d(q, p_{j+1})\}$.

THEOREM 2. *If S is a sparse point set, then $CL(S) \subseteq MWT(S)$.*

Proof. Let $CL(S) = (C_1, \dots, C_k)$. Clearly, the convex hull of S, C_1, is in $MWT(S)$. We shall first prove that C_2 is in $MWT(S)$ by contradiction. That is, if the edge set of C_2 contains a subset E which does not belong to $MWT(S)$, then we can construct a new triangulation $T(S)$ which contains all the edges of C_2 and has a weight less than $\omega(MWT(S))$. After proving that C_2 belongs to $MWT(S)$, we can remove all the vertices of C_1 from S since none of them will affect the minimum

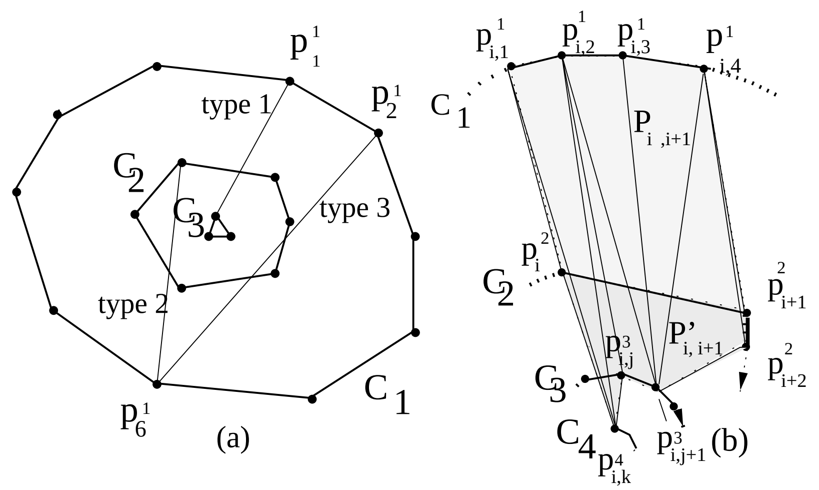

Figure 3. An illustration of three types of edges and an example of induced polygons.

weight triangulation of the remaining vertices in $R(C_2)$. Thus, we can recursively apply the same proof method to $S/\{V(C_1)\}$ until the proof is completed.

For clarity, we use superscript i to denote the vertices of the ith convex layer and use a subscript to denote the ordering of these vertices in that layer. In C_2, let $E = \{\overline{p_1^2 p_2^2}, \overline{p_2^2 p_3^2}, \dots, \overline{p_r^2 p_{r+1}^2}\}$ be the edge set not belonging to $MWT(S)$, where the vertices $(p_1^2, p_2^2, \dots, p_{r+1}^2\}$ are reindexed in clockwise order around C_2 since some edges of C_2 may not belong to E. Let $\bar{E}$ be the set of edges in $MWT(S)$, each of which crosses an element of E. There are three possible types of edges in $\bar{E}$ as shown in Figure 3(a). We delete $\bar{E}$ from the edge set of $MWT(S)$ and obtain a graph $MWT(S)/\bar{E}$. There are two cases w.r.t. this graph: (a) $\bar{E}$ does not contain any type-3 edge and (b) $\bar{E}$ contains some type-3 edges. We shall discuss these two cases separately.

In case (a), let $\bar{E}_i$ denote the subset of $\bar{E}$ crossing $\overline{p_i^2 p_{i+1}^2}$. Note that all the endpoints of $\bar{E}_i$ ending at C_1 together with p_i^2 and p_{i+1}^2 form a convex polygon $P_{i,i+1}$, and all these endpoints of $\bar{E}_i$ inside $R(C_2)$ together with p_i^2 and p_{i+1}^2 form a polygon $P'_{i,i+1}$. That is, $P_{i,i+1} \cup P'_{i,i+1}$ is the polygon triangulated by the edges of $\bar{E}_i$. Clearly, any two such polygons: $P_{i,i+1} \cup P'_{i,i+1}$ and $P_{j,j+1} \cup P'_{j,j+1}$ for $i \neq j$ and $i, j \epsilon [1, r]$ are disjoint since these edges in $\bar{E}_i$ (crossing $\overline{p_i^2 p_{i+1}^2}$) and those in $\bar{E}_j$ (crossing $\overline{p_j^2 p_{j+1}^2}$) must belong to the same $MWT(S)$. In particular, we re-index the vertices of $P_{i,i+1}$ as $(p_i^2, p_{i,1}^1, p_{i,2}^1, \dots, p_{i,k_i}^1, p_{i+1}^2, p_i^2)$. In general, these polygons will be $P_{1,2} = (p_1^2, p_{1,1}^1, p_{1,2}^1 \cdots, p_{1,k_1}^1, p_2^2, p_1^2)$; $P_{2,3} = (p_2^2, p_{2,1}^1, p_{2,2}^1 \cdots, p_{2,k_2}^1, p_3^2, p_2^2)$; $\cdots$; $P_{r,r+1} = (p_r^2, p_{r,1}^1, p_{r,2}^1, \cdots, p_{r,k_r}^1, p_{r+1}^2, p_r^2)$. Clearly, they are convex polygons lying outside C_2 and inside C_1. Refer to Figure 3(b), where $P_{i,i+1} = (p_i^2, p_{i,1}^1, p_{i,2}^1, p_{i,3}^1, p_{i,4}^1, p_{i+1}^2, p_i^2)$, and $P'_{i,i+1} = (p_i^2, p_{i+1}^2, p_{i+2}^2, p_{i,j+1}^3, p_{i,j}^3, p_{i,k}^4,$

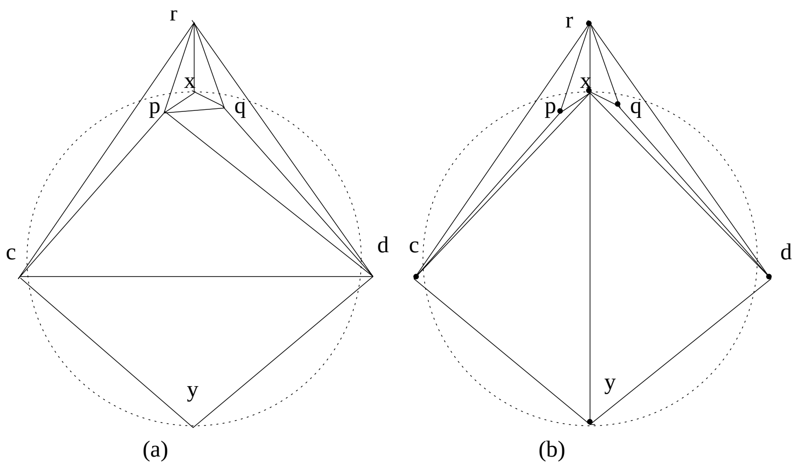

Figure 4. Two disjoint polygons.

p_i^2). Also refer to Figure 4, where $P_{i,i+1} \cup P'_{i,i+1}$ and $P_{i+j,i+j+1} \cup P'_{i+j,i+j+1}$ are disjoint.

Now, we re-triangulate the interior of $P_{i,i+1} \cup P'_{i,i+1}$ by an edge set $E_i = \{\overline{p_i^2 p_{i+1}^2}, \overline{p_i^2 p_{i,2}^1}, \overline{p_i^2 p_{i,3}^1}, \ldots, \overline{p_i^2 p_{i,k_i}^1}, E''\}$, where E'' is the subset of E_i that triangulates $P'_{i,i+1}$. This is always doable since $P_{i,i+1}$ is a convex polygon and since E'' has no special restriction. By Fact 3, $|\ \bar{E}_i\ |=|\ E_i\ |$ since the two edge sets respectively are the internal edge sets of two different triangulations for the same polygon. There is a one-to-one correspondence between the edges of $\bar{E}_i$ and the edges of E_i. Now, first match each edge of $\overline{p_i^2 p_{i,j}^1}$ for $2 \leqslant j \leqslant k_i$ with an edge of $\bar{E}_i$ ending at $p_{i,j}^1$. If only $p_{i,1}^1$ exists, then do only the subsequent matching. Match then $\overline{p_i^2 p_{i+1}^2}$ and E'' with the remaining edges in $\bar{E}_i$ in an arbitrary manner. By Condition (ii), each edge in $\overline{p_i^2 p_{i,j}^1}$ for $2 \leqslant j \leqslant k_i$ is shorter than the corresponding edge in $\bar{E}_i$, and by Condition (i), each edge in $E'' \cup \{\overline{p_i^2 p_{i+1}^2}\}$ (which cannot be longer than the diameter of $R(C_2)$) is also shorter than the corresponding edge in $\bar{E}_i$ (which is longer than the diameter of $R(C_2)$). Thus, the new triangulation for $P_{i,i+1} \cup P'_{i,i+1}$ with interior edge set E_i has less weight than the old triangulation with internal edge set $\bar{E}_i$. Consequently, we obtain a triangulation $T(S)$ with weight less than $\omega(MWT(S))$, a contradiction.

In case (b), a type-3 edge must cross two polygons in the area $R_{1,2}$. For example, in Figure 5 type-3 edge $\overline{p_{i,1}^1 p_{i+j,2}^1}$ crosses both $P_{i,i+1}$ and $P_{i+j,i+j+1}$. However, such a type-3 edge cannot exist in any $MWT(S)$. To see this, note that a type-3 edge or a group of neighboring type-3 edges induce a convex polygon in $MWT(S)$. Two of the vertices of this convex polygon must not belong to C_1, say p^s and p^t, and the remaining vertices must belong to C_1. By condition (i), these remaining vertices must lie outside the circles with radius $\overline{p^s p^t}$ and with center p^s or p^t. Then, by

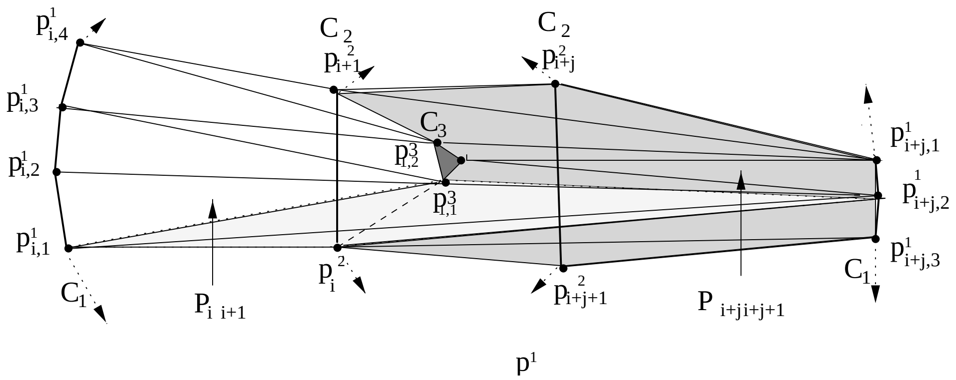

Figure 5. The lightly shaded quadrilateral is shared by $P_{i,i+1} \cup P'_{i,i+1}$ and $P_{i+j,i+j+1} \cup P'_{i+j,i+j+1}$. The heavily shaded triangle does not belong to any one of them.

[21], $\overline{p^s p^t}$ must belong to any $MWT(S)$, hence such type-3 edges cannot exist. We conclude that E belongs to $MWT(S)$, thus C_2 belongs to $MWT(S)$. Obviously, C_2 separates the vertices in C_1 from those in $R(C_2)$ in the MWT of S.

By removing all the vertices of C_1 from S, we have an original problem with one less convex layer. The above argument can be applied to $CL(S/V(C_1)) = (C_2, \ldots, C_k)$ and results in $C_3 \in MWT(S)$. We then remove all the vertices of C_2 from S and obtain an original problem with two less convex layers. This proof continues until $CL(S/(V(C_1) \cup \ldots \cup V(C_{k-1}))) = C_k$. Then, $C_k \in MWT(S)$ must hold. □

Generally speaking, $MWT_{CL}(S)$ is not an $MWT(S)$. Figure 1 illustrates a point set S such that $MWT(S) \neq MWT_{CL}(S)$. But from Theorem 1 and Theorem 2, we have that

THEOREM 3. *If S is a sparse point set, then $MWT_{CL}(S) = MWT(S)$ and the $MWT(S)$ can be computed in $O(n^4)$ times.*

5. Concluding remarks

In this paper, we presented an $O(n^4)$ algorithm for computing an $MWT(S)$ of sparse point set S with n elements. We may regard point set S with constraints and MWT of S with some predetermined edges as being a natural extension of the general MWT of S. For example, forcing the boundary of a simple polygon P to be in any $MWT(V(P))$ is a well-known constraint [13]. Convex-layers constraint seems to be a reasonable extension in this direction. It is quite interesting to find other constraints for MWT. Another example is restricting point set S to be on k convex layers [1] or to be on k non-intersecting straight line segments in $CH(S)$ [15]. A sparse point set is also such an example.

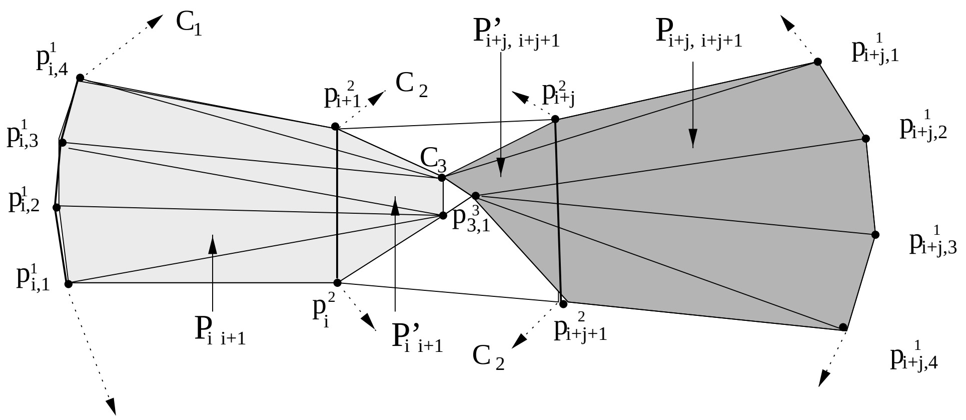

Figure 6.

Furthermore, by the analysis of computing an $MWT(S)$ of a sparse point set S, we can derive a sufficient condition for new subgraphs of MWT.

Sufficient condition

Let $CL(S) = (C_1, C_2, \ldots, C_k)$ be the convex layers of a point set S. Convex layers C_i for $1 < i \leqslant k$ belongs to an $MWT(S)$ if the following conditions are satisfied:

(i) $d(V(C_s), V(C_{s+1})) \geqslant D(V(C_{s+1}))$, for all $s = 1, \cdots i - 1$, and
(ii) if $\overline{p_s p_{s+1}}$ crosses $\overline{pq}$ for $\overline{p_s p_{s+1}} \in C_s$, $p, q \in S$, and $p \in C_j$ for $1 \leqslant j < s \leqslant i - 1$, then $d(p, q) > \max\{d(p, p_s), d(p, p_{s+1})\}$.

The new subgraph (if it exists) is totally different from the known subgraphs given in [5, 7, 9, 12, 18, 21]. Figure 6(a) gives an example showing that our new subgraph is different from all the known subgraphs of [9, 12, 19, 21], where $\overline{pq}$ can be found by our method but $\overline{pq}$ does not belong to the subgraphs identified by any other method mentioned above. Clearly, x lies inside the empty disk associated with $\overline{pq}$ in Keil's β-skeleton and x also lies inside the empty double-circle in the condition of [21]. $\overline{pq}$ is not the shortest edge among the seven points, thus, it cannot be found according to [9]. $\overline{pq}$ is not a stable edge in [19]. Figure 6(b) shows that $\overline{pq}$ cannot be in $T_4(S)$ of [7] since $\overline{xy}$ belongs to a local optimal triangulation as shown.

It is interesting to see some experimental result based on our result.

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